On the continuity of functions

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Abstract. Some theorems on continuity are presented. First we will prove that every convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous using nonstandard analysis methods. Then we prove that if the image of every compact (resp. convex) is compact (resp. convex), then the function is continuous.

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1 Sufficient Conditions for Continuity

The purpose of this paper is to present some results on continuity. Now let us introduce some terminology. In what follows, if E is a (standard) set, ${}^*E$ will denote its nonstandard extension. If $(E, |\cdot|)$ is a normed space and $x, y \in {}^*E$, we say that $x \approx y$ if $x - y$ is infinitesimal, *i.e.*, if $|x - y| < r$ for all positive real $r \in \mathbb{R}$; if x is standard and $x \approx y$, we say that y is near-standard and write $x = st(y)$. For further details, the reader is referred to [3], [4], [5] or [6].

Definition 1 Let E be a linear space and consider a function $f : E \to \mathbb{R}$. The function f is called convex if

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \text{ (Jensen's inequality)} \tag{1.1}$$

for all $x_1, x_2 \in E$ and $\lambda \in]0, 1[$.

Theorem 1 *Let $(E, |\cdot|)$ be a normed space and $f : E \to \mathbb{R}$ a convex function. If $f({}^*S^1) \subseteq fin({}^*\mathbb{R})$, where S^1 denotes the unit sphere in E and $fin({}^*\mathbb{R})$ the set of finite hyperreals, then f is continuous.*

Proof. Fix any $x_0 \in E$. Without any loss of generality, we may assume that $x_0 = 0$ and $f(x_0) = 0$ (simply replace f by the convex function $g(x) := f(x + x_0) - f(x_0)$). Then given $0 \approx \epsilon \in {}^*E$, $\epsilon \neq 0$, we have that

1. $f(\epsilon) \lesssim 0$ because

$$(1.2)\quad f(\epsilon) = f\left((1-|\epsilon|\,)0 + |\epsilon| \cdot \frac{\epsilon}{|\epsilon|}\right) \leq (1-|\epsilon|\,)f(0) + |\epsilon| \cdot f\left(\frac{\epsilon}{|\epsilon|}\right) \approx 0.$$

2. $f(\epsilon) \gtrsim 0$ because

$$(1.3)\qquad 0 = \frac{1}{1+|\epsilon|}\epsilon + \frac{|\epsilon|}{1+|\epsilon|} \cdot \frac{-\epsilon}{|\epsilon|}$$

   and so

$$(1.4)\qquad 0 \leq \frac{1}{1+|\epsilon|} f(\epsilon) + \frac{|\epsilon|}{1+|\epsilon|} f\left(\frac{-\epsilon}{|\epsilon|}\right) \Rightarrow f(\epsilon) \geq -|\epsilon| \cdot f\left(\frac{-\epsilon}{|\epsilon|}\right) \approx 0.$$

We conclude then that $f(\epsilon) \approx 0$. ∎

We will now see the special case when E is a finite dimensional space. First we need the following result due to Michel Goze (see [1] or [2]):

Theorem 2 *Let $M \in {}^*\mathbb{R}^n$ be an infinitesimal vector. Then there are non-null infinitesimals $\epsilon_1, \ldots, \epsilon_k \in {}^*\mathbb{R}$ and standard vectors $V_1, \ldots, V_k \in \mathbb{R}^n$, for some $k \leq n$, with*

$$(1.5)\qquad M = \epsilon_1 V_1 + \epsilon_1\epsilon_2 V_2 + \ldots + \epsilon_1\epsilon_2 \ldots \epsilon_k V_k.$$

With this we can prove the well known theorem:

Theorem 3 *Every convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.*

Proof. Again we assume that $x_0 = 0$ and $f(x_0) = 0$. Fix any $\epsilon \approx 0$ and write $\epsilon = \epsilon_1 V_1 + \epsilon_1\epsilon_2 V_2 + \ldots + \epsilon_1\epsilon_2 \ldots \epsilon_k V_k$. We can also assume that all the infinitesimals ϵ_i are positive (replacing V_i by $-V_i$ if necessary).

1. $f(\epsilon) \lesssim 0$:

$$(1.6)\qquad f(\epsilon) = f\left((1-\epsilon_1)0 + \epsilon_1(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k)\right) \leq$$
$$(1-\epsilon_1)f(0) + \epsilon_1 f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k).$$

   It is enough to prove that $f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k)$ is bounded from above:

$$(1.7)\qquad f(V_1 + \epsilon_2 V_2 + \epsilon_2\epsilon_3 V_3 + \ldots + \epsilon_2\epsilon_3 \ldots \epsilon_k V_k) =$$
$$f\left((1-\epsilon_2)V_1 + \epsilon_2(V_1 + V_2 + \epsilon_3 V_3 + \ldots + \epsilon_3 \ldots \epsilon_k V_k)\right) \leq$$

$$(1-\epsilon_2)f(V_1)+\epsilon_2 f(V_1+V_2+\epsilon_3 V_3+\ldots+\epsilon_3\ldots\epsilon_k V_k).$$

To see that $f(V_1+V_2+\epsilon_3 V_3+\ldots+\epsilon_3\ldots\epsilon_k V_k)$ is bounded above, we have

$$f(V_1+V_2+\epsilon_3 V_3\ldots+\epsilon_3\ldots\epsilon_k V_k)= \tag{1.8}$$

$$f\left((1-\epsilon_3)(V_1+V_2)+\epsilon_3(V_1+V_2+V_3+\epsilon_4 V_4+\ldots.+\epsilon_4\ldots\epsilon_k V_k)\right)\leq$$

$$(1-\epsilon_3)f(V_1+V_2)+\epsilon_3 f(V_1+V_2+V_3+\epsilon_4 V_4+\ldots+\epsilon_4\ldots\epsilon_k V_k).$$

Repeating this process we obtain

$$f(V_1+V_2+\ldots+\epsilon_k V_k)\leq(1-\epsilon_k)f(V_1+V_2+\ldots+V_{k-1})+\epsilon_k f(V_1+V_2+\ldots+V_k) \tag{1.9}$$

which is bounded from above.

2. $f(\epsilon)\gtrsim 0$:

   Since

$$0=\frac{1}{1+\epsilon_1}\epsilon+\frac{\epsilon_1}{1+\epsilon_1}\cdot\frac{-\epsilon}{\epsilon_1} \tag{1.10}$$

   we obtain

$$0\leq\frac{1}{1+\epsilon_1}f(\epsilon)+\frac{\epsilon_1}{1+\epsilon_1}f\left(\frac{-\epsilon}{\epsilon_1}\right)\Rightarrow f(\epsilon)\geq-\epsilon_1 f\left(\frac{-\epsilon}{\epsilon_1}\right). \tag{1.11}$$

   If

$$f\left(\frac{-\epsilon}{\epsilon_1}\right)=f(-V_1-\epsilon_2 V_2-\ldots-\epsilon_2\ldots\epsilon_k V_k) \tag{1.12}$$

   is bounded from above then $f(\epsilon)\gtrsim 0$. Replacing V_i by $W_i:=-V_i$, with the same calculations as presented before, we conclude the desired.

■

For our next result, we need the following: If A is a compact set, then for each $a\in{}^*A$, there exists $st(a)$ and $st(a)\in A$.

Theorem 4 *Let $(E,|\cdot|)$ be a finite-dimensional normed space, $(F,\mathcal{T})$ a Hausdorff linear topological space and $f:E\to F$ a function. If the image of every compact subspace of E is compact in F and the image of every convex subspace of E is convex in F, then f is continuous.*

Proof. Fix $x\in E$ and $y\in{}^*E$ with $y\approx x$. For every $n\in\mathbb{N}$, the closed ball $\overline{B_{1/n}(x)}$ is compact and convex, so $F_n:=f\left(\overline{B_{1/n}(x)}\right)$ is also compact and convex. Besides this, we have for each $n\in\mathbb{N}$

$$x,y\in{}^*\overline{B_{1/n}(x)}\Rightarrow f(x),f(y)\in{}^*F_n\Rightarrow f(x),st(f(y))\in F_n. \tag{1.13}$$

So there exists

$$x_n \in \overline{B_{1/n}(x)} \text{ with } f(x_n) = \frac{1}{n} f(x) + \left(1 - \frac{1}{n}\right) st(f(y)). \tag{1.14}$$

Since $\lim x_n = x$, the set $A := \{x\} \cup \{x_n | n \in \mathbb{N}\}$ is compact and so $f(A) = \{f(x_n) | n \in \mathbb{N}\}$ is also compact. Consequently, $f(x) = st(f(y))$. ■

As a consequence, we have:

Theorem 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If the image of every compact subset of $\mathbb{R}$ is compact and the image of every connected subset of $\mathbb{R}$ is connected, then f is continuous.*

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